# Power Series

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## Exercises

### The Fibonacci Sequence

We search for a closed form of the Fibonacci sequence $a_n$, defined by

$$a_0 = 0,\ a_1 = 1,\ \forall\, n \in \mathbb{N},\ a_{n+2} = a_n + a_{n+1}.$$

#### Questions

1. Show that the golden ratio

$$\phi = \frac{1+\sqrt{5}}{2}$$

is the largest solution of the equation $x^2 = x+1$ and that the other solution is $\psi = -1/\phi$.

2. Establish that for any $n \in \mathbb{N}$, $a_n \leq \phi^n$.

3. Show that the radius of convergence of the generating function

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n$$

is at least $1/\phi$.

4. Compute $f(z)$ when $|z| < 1/\phi$.

5. Find a closed form for $a_n$, $n \in \mathbb{N}$.

**Answers**

1. The discriminant $\Delta$ of the quadratic equation $x^2 - x - 1 = 0$ is

$$\Delta = (-1)^2 - 4 \times 1 \times (-1) = 5,$$

therefore the solutions are

$$x = \frac{1 \pm \sqrt{5}}{2}.$$

The golden ratio $\phi$, equal to $(1+\sqrt{5})/2$, is the largest of the two. The fact that the other root $\psi$ of the equation is equal to $-1/\phi$ can be demonstrated directly; we have indeed

$$\psi = \frac{1-\sqrt{5}}{2} = \frac{1+\sqrt{5}}{1+\sqrt{5}}\frac{1-\sqrt{5}}{2} = \frac{1^2 - \sqrt{5}^2}{2(1+\sqrt{5})} = -\frac{2}{1+\sqrt{5}}.$$

Alternatively, we know that

$$x^2 - x - 1 = (x-\phi)(x-\psi) = x^2 - (\phi+\psi)x + \phi\psi,$$

hence $\phi\psi = -1$.

2. It is clear that $a_0 = 0 \leq 1 = \phi^0$ and $a_1 = 1 \leq \phi = \phi^1$. If we assume that the inequality $a_n \leq \phi^n$ holds for $n = 0, 1, \ldots, m+1$, the recursive definition of the Fibonacci sequence yields

$$a_{m+2} = a_m + a_{m+1} \leq \phi^m + \phi^{m+1} = \phi^m(1+\phi) = \phi^{m+2}.$$

Hence, by induction, the inequality holds for every $n \in \mathbb{N}$.

3. The inequality $a_n \leq \phi^n$ provides

$$\limsup_{n\to+\infty} \sqrt[n]{|a_n|} \leq \phi,$$

and hence, by the Cauchy-Hadamard formula, the radius of convergence of the series $\sum_{n\geq 0} a_n z^n$ is at least $1/\phi$.

4. If $|z| < 1/\phi$, we can write the expansion of $f(z)$ as

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n = a_0 + a_1 z + \sum_{n=0}^{+\infty} a_{n+2} z^{n+2} = z + \sum_{n=0}^{+\infty} a_{n+2} z^{n+2}.$$

Using $a_{n+2} = a_n + a_{n+1}$, we deduce that

$$f(z) = z + z^2 \sum_{n=0}^{+\infty} a_n z^n + z \sum_{n=0}^{+\infty} a_{n+1} z^{n+1} = z + z^2 f(z) + z f(z),$$

hence

$$f(z) = \frac{z}{1 - z - z^2}.$$

5. The roots of the polynomial $1 - z - z^2$ are $-\phi$ and $-\psi$, hence

$$-z^2 - z + 1 = -(z + \phi)(z + \psi).$$

Thus, for any $|z| < 1/\phi$, we have

$$f(z) = \frac{-z}{(z+\phi)(z+\psi)} = \frac{1}{\phi - \psi}\left[\frac{-\phi}{z+\phi} + \frac{\psi}{z+\psi}\right],$$

or equivalently, using $\psi = -1/\phi$,

$$f(z) = \frac{1}{\phi - \psi}\left[\frac{-1}{1-\psi z} + \frac{1}{1-\phi z}\right].$$

If $|z| < 1/\phi$, then $|\phi z| < 1$ and $|\psi z| < 1$ and consequently

$$\frac{1}{1-\phi z} = \sum_{n=0}^{+\infty} \phi^n z^n, \ \frac{1}{1-\psi z} = \sum_{n=0}^{+\infty} \psi^n z^n.$$

Thus, $f(z)$ can be expanded as

$$f(z) = \sum_{n=0}^{+\infty} \frac{1}{\phi - \psi}\left[\phi^n - \psi^n\right] z^n.$$

The power series expansion of $f(z)$ in the disk centered on the origin with radius $1/\phi$ is unique, therefore

$$a_n = \frac{1}{\phi - \psi}\left[\phi^n - \psi^n\right] = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

for every $n \in \mathbb{N}$.

## Entire Functions Dominated By Polynomials

### Question

Show that if a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is dominated by a polynomial $P$ of order $p$

$$\forall\, z \in \mathbb{C},\ |f(z)| \leq |P(z)|$$

then it is a polynomial whose degree is at most $p$.

### Answer

Let $\sum_{n=0}^{+\infty} a_n z^n$ be the power series expansion of $f$ in $\mathbb{C}$. For any $r > 0$, we have

$$a_n = \frac{1}{i2\pi} \int_{r[\circlearrowleft]} \frac{f(z)}{z^{n+1}} dz,$$

hence by the M-L estimation lemma,

$$|a_n| \leq \frac{\sup\{|P(re^{i2\pi t})| \mid t \in [0,1]\}}{r^n}.$$

For any $n > p$, letting $r \to +\infty$ provides $a_n = 0$. Hence, the function $f$ is a polynomial of degree at most $p$.

## Existence of Primitives

### Question

Show that the function

$$f : z \in \mathbb{C} \setminus [-1,1] \mapsto \frac{\pi}{z} \frac{1}{\sin \pi/z}$$

has a primitive.

### Answer

The function $f$ is defined and holomorphic in $\mathbb{C} \setminus [-1,1]$ (the zeros of $\sin \pi/z$ are $z = 1/k$ for $k \in \mathbb{N}^*$).

We first consider the restriction of $f$ to the annulus $A(0,1,+\infty)$. For any $z$ in this annulus, $-z$ also belong to it and $f(-z) = f(z)$. Hence, if $\sum_{n=-\infty}^{+\infty} a_n z^n$ is a Laurent series expansion of $f$, $\sum_{n=-\infty}^{+\infty} (-1)^n a_n z^n$ is another valid one. The

uniqueness of the expansion yields that $a_n = 0$ if $n$ is odd; in particular, $a_{-1} = 0$ and the sum

$$\sum_{p=-\infty}^{+\infty} \frac{a_{2p}}{2p+1} z^{2p+1}$$

provide a primitive of $f$ on the annulus.

Now, let $\gamma$ be an arbitrary closed rectifiable path of $\mathbb{C} \setminus [-1,1]$. Let $n = \mathrm{ind}(\gamma, 0)$; define the path $\mu : t \in [0,1] \mapsto 2e^{i2\pi nt}$ and the sequence of paths $\nu = (\gamma, \mu^{\leftarrow})$. As $[-1,1]$ is a connected subset of $\mathbb{C} \setminus \nu([0,1])$, for any $z \in [-1,1]$, $\mathrm{ind}(\nu, z) = \mathrm{ind}(\nu, 0) = 0$. Consequently, $\mathrm{Int}\, \nu \subset \mathbb{C} \setminus [-1,1]$ and Cauchy's integral theorem provides

$$\int_\gamma f(z)\, dz = \int_\mu f(z)\, dz.$$

As $f$ has a primitive on the annulus $A(0,1,+\infty)$, the integral in the right-hand side of this equation is equal to zero. The classic criteria therefore proves that primitives of $f$ exist in $\mathbb{C} \setminus [-1,1]$.

## A Removable Set

Let $f : \mathbb{C} \to \mathbb{C}$ be a continuous function which is holomorphic on $\mathbb{C} \setminus \mathbb{U}$ (where $\mathbb{U} = \{z \in \mathbb{C} \mid |z| = 1\}$).

### Question

Prove that $f$ is an entire function.

### Answer

Let $\sum_{n=0}^{+\infty} a_n z^n$ be the Taylor series expansion of $f$ in $D(0,1)$; we are going to prove that this expansion is actually a valid expansion of $f$ in $\mathbb{C}$. Consider the Laurent expansion $\sum_{n=-\infty}^{+\infty} b_n z^n$ of $f$ in $A(0,1,+\infty)$. For any $n \in \mathbb{Z}$ and any $r > 1$, we have

$$b_n = \frac{1}{i2\pi} \int_{r[\circlearrowleft]} \frac{f(z)}{z^{n+1}} dz,$$

thus, by continuity of $f$

$$\begin{aligned} b_n &= \lim_{r \to 1^+} \frac{1}{i2\pi} \int_{r[\circlearrowleft]} \frac{f(z)}{z^{n+1}} dz \\ &= \lim_{r \to 1^-} \frac{1}{i2\pi} \int_{r[\circlearrowleft]} \frac{f(z)}{z^{n+1}} dz \end{aligned}$$

and consequently, $b_n = a_n$ if $n$ is non-negative and zero otherwise. The sum $\sum_{n=0}^{+\infty} a_n z^n$ is defined for any $|z| > 1$, thus its open disk of convergence is the

full complex plane. It is equal to $f$ on $\mathbb{C} \setminus \mathbb{U}$ and both functions are continuous on $\mathbb{C}$, hence they are equal on $\mathbb{C}$: the function $f$ is entire.

## Derivative of Power Series

### Question

Provide an alternate proof of the existence and value of the derivative of the sum $\sum_{n=0}^{+\infty} a_n(z-c)^n$ in its open disk of convergence.

Hint: a locally uniform limit of a sequence of holomorphic functions is holomorphic.

### Answer

Let $f_m(z) = \sum_{n=0}^{m} a_n(z-c)^n$. Every polynomial $f_m$ is holomorphic and the sequence converges locally uniformly to $f(z) = \sum_{n=0}^{+\infty} a_n(z-c)^n$ in the open disk of convergence $D(c,r)$ of the series, thus $f$ is holomorphic.

For any holomorphic function $\phi$ in $D(c,r)$ and any $\rho \in ]0,r[$

$$\phi'(z) = \frac{1}{i2\pi} \int_{c+\rho[\circlearrowleft]} \frac{\phi(w)}{(w-z)^2}.$$

Thus, for any $m \in \mathbb{N}$,

$$f_m'(z) = \sum_{n=1}^{m} n a_n (z-c)^{n-1} = \frac{1}{i2\pi} \int_{c+\rho[\circlearrowleft]} \frac{f_m(w)}{(w-z)^2}.$$

The integrand above converges locally uniformly in $D(c,r)$, hence

$$\lim_{m\to+\infty} \frac{1}{i2\pi} \int_{c+\rho[\circlearrowleft]} \frac{f_m(w)}{(w-z)^2} = \frac{1}{i2\pi} \int_{c+\rho[\circlearrowleft]} \frac{f(w)}{(w-z)^2} = f'(z).$$

Finally,

$$\sum_{n=1}^{+\infty} n a_n (z-c)^{n-1} = \lim_{m\to+\infty} \sum_{n=1}^{m} n a_n (z-c)^{n-1} = f'(z).$$