

Line Integrals & Primitives

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Introduction

The main goal of this chapter is to derive the fundamental theorem of calculus for functions of a complex variable. This theorem characterizes the relation between functions and their primitives with the help of integrals. A version of this theorem for functions of a real variable is the following:

Theorem – Fundamental Theorem of Calculus (Real Analysis). Let I be an open interval of \mathbb{R} , $f : I \rightarrow \mathbb{R}$ be a continuous function and $a \in I$. A function $g : I \rightarrow \mathbb{R}$ is a primitive of f if and only if it satisfies

$$\forall x \in I, g(x) = g(a) + \int_a^x f(t) dt.$$

Proof. Suppose that the function g satisfies the integral equation of the theorem.

For any $x \in I$ and any real number h such that $x + h \in I$,

$$\begin{aligned} \frac{g(x+h) - g(x)}{h} &= \frac{1}{h} \int_x^{x+h} f(t) dt \\ &= \frac{1}{h} \int_x^{x+h} f(x) dt + \frac{1}{h} \int_x^{x+h} (f(t) - f(x)) dt \\ &= f(x) + \frac{1}{h} \int_x^{x+h} (f(t) - f(x)) dt, \end{aligned}$$

Let $\epsilon > 0$; by continuity of f at x , there is a $\delta > 0$ such that

$$\forall t \in I, (|t - x| \leq \delta \Rightarrow |f(t) - f(x)| < \epsilon)$$

thus if $|h| < \delta$,

$$\left| \frac{g(x+h) - g(x)}{h} - f(x) \right| \leq \frac{1}{|h|} |h| \times \epsilon = \epsilon.$$

The difference quotient tends to $f(x)$ when h tends to zero: $g'(x)$ exists and is equal to $f(x)$.

Conversely, suppose that $e : I \rightarrow \mathbb{R}$ is a primitive of f . The difference d between e and the function

$$g : x \in I \mapsto e(a) + \int_a^x f(t) dt$$

is zero at a and has a zero derivative on I . By the mean value theorem, for any $x \in I$ such that $x \neq a$, there is a $b \in I$ such that

$$\frac{d(x) - d(a)}{x - a} = d'(b) = 0,$$

hence $d(x) = d(a) = 0$ and therefore $e = g$. ■

Paths

Definition – Path. A *path* γ is a continuous function from $[0, 1]$ to \mathbb{C} . If A is a subset of the complex plane, γ is a *path of* A if additionally $\gamma([0, 1]) \subset A$.

Definition – Image of a Path. The *image* or *trajectory* or *trace* of the path γ is the image $\gamma([0, 1])$ of the interval $[0, 1]$ by the function γ .

Definition – Path Endpoints. The complex numbers $\gamma(0)$ and $\gamma(1)$ are the *initial point* and *terminal point* of γ – they are its *endpoints*; the path γ *joins* its initial and terminal points. The path is *closed* if the initial and terminal point are the same. The paths $\gamma_1, \dots, \gamma_n$ are *consecutive* if for $k = 1, \dots, n - 1$, the terminal point of γ_k is the initial point of γ_{k+1} .

Example – Oriented Line Segment. The oriented line segment (or simply oriented segment) with initial point $a \in \mathbb{C}$ and terminal point $b \in \mathbb{C}$ is denoted $[a \rightarrow b]$ and defined as

$$[a \rightarrow b] : t \in [0, 1] \mapsto (1 - t)a + tb.$$

Its image is the line segment $[a, b]$.

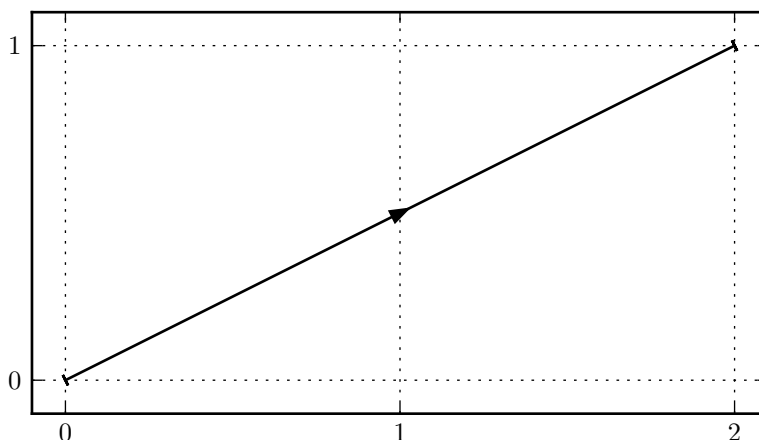


Figure 1: Representation of the oriented line segment $[0 \rightarrow 2 + i]$

Example – Oriented Circle. The oriented circle of radius one centered at the origin traversed once in the positive sense (counterclockwise) is denoted $[\odot]$ and defined as

$$[\odot] : t \in [0, 1] \rightarrow e^{i2\pi t}.$$

The circle of radius $r \geq 0$ centered at $c \in \mathbb{C}$ traversed $n \in \mathbb{Z}^*$ times in the positive sense is the path

$$c + r[\odot]^n : t \in [0, 1] \rightarrow c + re^{i2\pi nt}.$$

Its image is the circle centered on c with radius r ; its initial and terminal points are both $c + r$, hence it is closed.

Definition – Open (Path-)Connected Sets. An open subset Ω of the complex-plane is *(path-)connected* if for any points x and y of Ω , there is a path of Ω that joins x and y .

Definition – Reverse Path. The *reverse* (or *opposite*) of the path γ is the path γ^{\leftarrow} defined by

$$\forall t \in [0, 1], \gamma^{\leftarrow}(t) = \gamma(1 - t).$$

Definition – Path Concatenation. Let $t_0 = 0 < t_1 < \dots < t_{n-1} < t_n = 1$ be a partition of the interval $[0, 1]$. The *concatenation* of consecutive paths

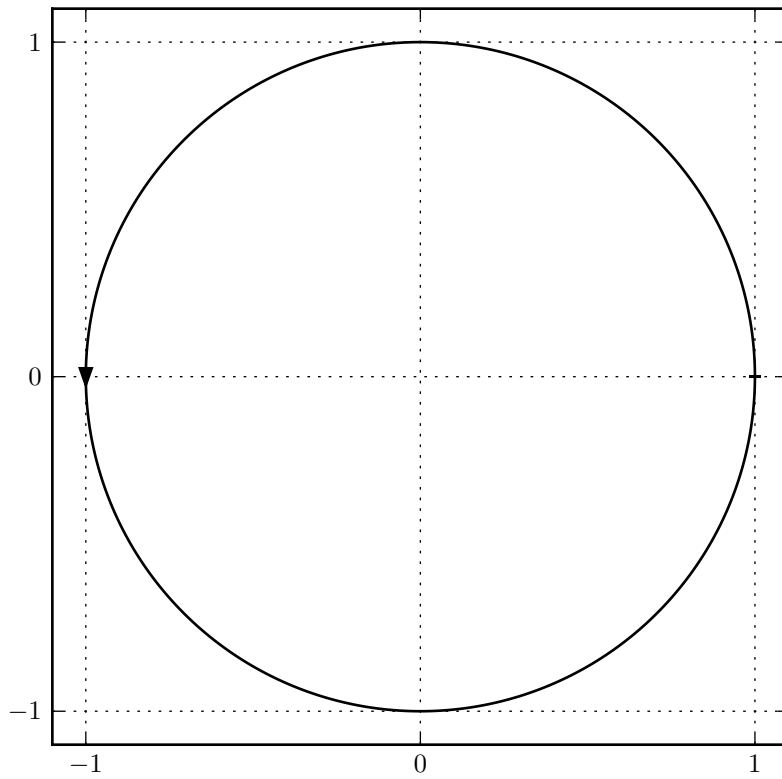


Figure 2: Representation of the oriented circle $[\odot]$

$\gamma_1, \dots, \gamma_n$ associated to this partition is the path γ denoted

$$\gamma_1 |_{t_1} \cdots |_{t_{n-1}} \gamma_n$$

such that

$$\forall k \in \{1, \dots, n\}, \gamma|_{[t_{k-1}, t_k]} = \gamma_k \left(\frac{t - t_{k-1}}{t_k - t_{k-1}} \right).$$

If the partition of $[0, 1]$ is uniform, that is, if

$$\forall k \in \{0, \dots, n\}, t_k = k/n,$$

we denote the concatenated path with the simpler notation

$$\gamma_1 | \cdots | \gamma_n.$$

Example – Oriented Polyline. An oriented polyline (or piecewise linear path) is the concatenation of consecutive oriented line segments. When the associated partition of $[0, 1]$ is uniform, we use the notation

$$[a_0 \rightarrow a_1 \rightarrow \cdots \rightarrow a_{n-1} \rightarrow a_n] = [a_0 \rightarrow a_1] | \cdots | [a_{n-1} \rightarrow a_n].$$

Definition – Rectifiable Path. A path $\gamma : [0, 1] \rightarrow \mathbb{C}$ is *rectifiable* if the function γ is piecewise continuous differentiable, that is, if there are consecutive continuously differentiable paths $\gamma_1, \dots, \gamma_n$ and a partition (t_0, \dots, t_n) of the interval $[0, 1]$ such that

$$\gamma = \gamma_1 |_{t_1} \cdots |_{t_{n-1}} \gamma_n.$$

We characterized initially connected sets via merely continuous paths. However, when such sets are open, we can use rectifiable paths instead:

Lemma – Connectedness & Rectifiable Paths. An open subset Ω of the complex plane is *connected* if and only if every pair of points of Ω may be joined by a rectifiable path of Ω .

Proof. If any pair of points of Ω can be joined by a rectifiable path of Ω , then Ω is connected. Conversely, assume that a (merely continuous) path γ of Ω joins x and y . Its image $\gamma([0, 1])$ is a compact subset of Ω – as the image of a compact set by a continuous function – thus the distance r between $\gamma([0, 1])$ and the closed set $\mathbb{C} \setminus \Omega$ is positive. Additionally, the function γ is uniformly continuous – as a continuous function with a compact domain of definition; there is a positive integer n such that

$$\forall t \in [0, 1], \forall s \in [0, 1], (|t - s| \leq 1/n \Rightarrow |\gamma(t) - \gamma(s)| < r).$$

For any $k \in \{0, \dots, n\}$, the point $\gamma(k/n)$ belongs to Ω ; the path μ defined as

$$\mu = [\gamma(0) \rightarrow \cdots \rightarrow \gamma(k/n) \rightarrow \cdots \rightarrow \gamma(1)]$$

is rectifiable and joins x and y . Now, for any $t \in [0, 1]$, let $k \in \{0, \dots, n-1\}$ be such that $t \in [k/n, (k+1)/n]$. We have

$$|\mu(t) - \gamma(k/n)| \leq |\gamma((k+1)/n) - \gamma(k/n)| < r,$$

therefore μ is a path of Ω . ■

Line Integrals

Definition – Length of a Rectifiable Path. The length of a continuously differentiable path γ is the nonnegative real number

$$\ell(\gamma) = \int_0^1 |\gamma'(t)| dt.$$

The length of a rectifiable path $\gamma = \gamma_1 |_{t_1} \cdots |_{t_{n-1}} \gamma_n$ – where every γ_k is continuously differentiable – is the nonnegative real number

$$\ell(\gamma) = \sum_{k=1}^n \ell(\gamma_k).$$

Example – Length of an Oriented Segment. The oriented segment $[a \rightarrow b]$ is continuously differentiable and thus rectifiable. For any $t \in [0, 1]$, $[a \rightarrow b]'(t) = b - a$, hence its length is

$$\ell([a \rightarrow b]) = \int_0^1 |b - a| dt = |b - a|.$$

Example – Length of an Oriented Circle. The oriented circle $c + r[\odot]^n$ centered at c with radius $r \geq 0$ traversed n times in the positive sense is continuously differentiable and thus rectifiable. For any $t \in [0, 1]$,

$$[c + r[\odot]^n]'(t) = (i2\pi n)re^{i2\pi nt},$$

hence the length of this path is

$$\ell(c + r[\odot]^n) = \int_0^1 |(i2\pi n)re^{i2\pi nt}| dt = \int_0^1 |2\pi nr| dt = 2\pi r \times |n|.$$

It differs from the length of its circle image – which is $2\pi r$ – unless the circle is traversed exactly once in the positive or negative sense.

Definition – Line Integral. The *line integral* along a continuously differentiable path γ of a complex-valued function f defined and continuous on the image of γ is the complex number defined by

$$\int_{\gamma} f(z) dz = \int_0^1 f(\gamma(t))\gamma'(t) dt.$$

If γ is the rectifiable path $\gamma_1 |_{t_1} \cdots |_{t_{n-1}} \gamma_n$ – where every γ_k is continuously differentiable – then the line integral along γ of f is defined as the sum of the line integrals of f along the γ_k :

$$\int_{\gamma} f(z) dz = \sum_{k=1}^n \int_{\gamma_k} f(z) dz.$$

Example – Integration along an Oriented Segment. The line integral of the continuous function $f : [a, b] \mapsto \mathbb{C}$ along the oriented segment $[a \rightarrow b]$ is

$$\begin{aligned} \int_{[a \rightarrow b]} f(z) dz &= \int_0^1 f((1-t)a + tb)(b-a) dt \\ &= (b-a) \int_0^1 f((1-t)a + tb) dt. \end{aligned}$$

Example – Integration along an Oriented Circle. The line integral of a continuous function $f : \{z \in \mathbb{C} \mid |z| = 1\} \rightarrow \mathbb{C}$ on the oriented circle $[\odot]$ is

$$\begin{aligned} \int_{[\odot]} f(z) dz &= \int_0^1 f(e^{i2\pi t})(i2\pi e^{i2\pi t} dt) \\ &= i \int_0^1 f(e^{i2\pi t}) e^{i2\pi t} (2\pi dt) \\ &= i \int_0^{2\pi} f(e^{i\theta}) e^{i\theta} d\theta. \end{aligned}$$

Theorem – M-L Inequality. For any rectifiable path γ and any continuous function $f : \gamma([0, 1]) \rightarrow \mathbb{C}$,

$$\left| \int_{\gamma} f(z) dz \right| \leq \left(\max_{z \in \gamma([0, 1])} |f(z)| \right) \times \ell(\gamma).$$

Proof. Let $\gamma_1 |_{t_1} \cdots |_{t_{n-1}} \gamma_n$ be a continuously differentiable decomposition of γ . For any $\ell \in \{1, \dots, n\}$,

$$\begin{aligned} \left| \int_{\gamma_{\ell}} f(z) dz \right| &= \left| \int_0^1 f(\gamma_{\ell}(t)) \gamma'_{\ell}(t) dt \right| \\ &\leq \int_0^1 |f(\gamma_{\ell}(t))| |\gamma'_{\ell}(t)| dt \\ &\leq \left(\max_{t \in [0, 1]} |f(\gamma_{\ell}(t))| \right) \times \int_0^1 |\gamma'_{\ell}(t)| dt \\ &= \left(\max_{z \in \gamma_{\ell}([0, 1])} |f(z)| \right) \times \ell(\gamma_{\ell}). \end{aligned}$$

Consequently,

$$\begin{aligned} \left| \int_{\gamma} f(z) dz \right| &\leq \sum_{\ell=1}^n \left| \int_{\gamma_{\ell}} f(z) dz \right| \\ &\leq \left(\max_{z \in \gamma([0, 1])} |f(z)| \right) \times \sum_{\ell=1}^n \ell(\gamma_{\ell}) \\ &\leq \left(\max_{z \in \gamma([0, 1])} |f(z)| \right) \times \ell(\gamma) \end{aligned}$$

which is the desired inequality. ■

A practical consequence of the M-L inequality:

Corollary – Convergence in Line Integrals. For any rectifiable path γ and any sequence of continuous function $f_n : \gamma([0, 1]) \rightarrow \mathbb{C}$ which converges uniformly to the function f , we have

$$\lim_{n \rightarrow +\infty} \int_{\gamma} f_n(z) dz = \int_{\gamma} f(z) dz.$$

Proof. The M-L inequality provides

$$\begin{aligned} \left| \int_{\gamma} f_n(z) dz - \int_{\gamma} f(z) dz \right| &= \left| \int_{\gamma} (f_n(z) - f(z)) dz \right| \\ &\leq \left(\max_{z \in \gamma([0,1])} |f_n - f(z)| \right) \times \ell(\gamma) \end{aligned}$$

which yields the desired result. ■

Theorem – Invariance By Reparametrization. Let $\gamma : [0, 1] \rightarrow \mathbb{C}$ be a continuously differentiable path. Let $\phi : [0, 1] \rightarrow [0, 1]$ be an increasing C^1 -diffeomorphism – a continuously differentiable function such that $\phi(0) = 0$, $\phi(1) = 1$ and $\phi'(t) > 0$ for any $t \in [0, 1]$. The following statements hold:

- The path $\mu = \gamma \circ \phi$ is a continuously differentiable path.
- It has the same initial point, terminal point and image as γ .
- The length of μ and γ are identical.
- For any continuous function $f : \gamma([0, 1]) \rightarrow \mathbb{C}$,

$$\int_{\mu} f(z) dz = \int_{\gamma} f(z) dz.$$

Proof. The function μ is continuously differentiable as the composition of continuously differentiable functions. We have

$$\mu(0) = \gamma(\phi(0)) = \gamma(0), \quad \mu(1) = \gamma(\phi(1)) = \gamma(1),$$

hence the endpoints of γ and μ are identical. The function ϕ is a bijection from $[0, 1]$ into itself, therefore

$$\mu([0, 1]) = \gamma(\phi([0, 1])) = \gamma([0, 1])$$

and the images of γ and μ are identical.

The length of μ is

$$\ell(\mu) = \int_0^1 |\mu'(t)| dt = \int_0^1 |\gamma'(\phi(t))\phi'(t)| dt = \int_0^1 |\gamma'(\phi(t))| \phi'(t) dt$$

The change of variable $s = \phi(t)$ provides

$$\int_0^1 |\gamma'(\phi(t))| \phi'(t) dt = \int_0^1 |\gamma'(s)| ds,$$

hence the lengths of γ and μ are equal. We also have

$$\int_{\mu} f(z) dz = \int_0^1 (f \circ \mu)(t) \mu'(t) dt = \int_0^1 (f \circ \gamma)(\phi(t)) \gamma'(\phi(t)) (\phi'(t) dt).$$

The same change of variable leads to

$$\int_{\mu} f(z) dz = \int_0^1 (f \circ \gamma)(s) \gamma'(s) ds = \int_{\gamma} f(z) dz,$$

which concludes the proof. ■

Definition – Image of a Path by a Function. Let $\gamma : [0, 1] \rightarrow \mathbb{C}$ be a path and $f : \gamma([0, 1]) \rightarrow \mathbb{C}$ be a continuous function. The *image of γ by f* is the path $f \circ \gamma$.

Theorem – Change of Variable. Let Ω be an open subset of \mathbb{C} , let γ be a rectifiable path of Ω and let $f : \Omega \rightarrow \mathbb{C}$ be a holomorphic function. The path $f \circ \gamma$ is rectifiable and for any continuous function $g : (f \circ \gamma)([0, 1]) \rightarrow \mathbb{C}$,

$$\int_{f \circ \gamma} g(z) dz = \int_{\gamma} g(f(w)) f'(w) dw.$$

Proof. Assume that $\gamma_1 |_{t_1} \dots |_{t_{n-1}} \gamma_n$ is a decomposition of γ into continuously differentiable paths. We have

$$f \circ \gamma = f \circ \gamma_1 |_{t_1} \dots |_{t_{n-1}} f \circ \gamma_n,$$

and for any $k \in \{1, \dots, n\}$, the function $f \circ \gamma_k$ is continuously (real-)differentiable with

$$(f \circ \gamma_k)'(t) = f'(\gamma_k(t)) \gamma_k'(t),$$

hence the path $f \circ \gamma$ is rectifiable. Moreover,

$$\int_{\gamma_k} g(f(w)) f'(w) dw = \int_0^1 g(f(\gamma_k(t))) f'(\gamma_k(t)) \gamma_k'(t) dt,$$

hence

$$\begin{aligned} \int_0^1 g(f(\gamma_k(t))) f'(\gamma_k(t)) \gamma_k'(t) dt &= \int_0^1 g(f(\gamma_k(t))) (f \circ \gamma_k)'(t) dt \\ &= \int_{f \circ \gamma_k} g(w) dw \end{aligned}$$

which proves the desired result. ■

Primitives

Definition – Primitive. Let $f : \Omega \rightarrow \mathbb{C}$ where Ω is an open subset of \mathbb{C} . A *primitive* (or *antiderivative*) of f is a holomorphic function $g : \Omega \rightarrow \mathbb{C}$ such that $g' = f$.

Theorem – Fundamental Theorem of Calculus (Complex Analysis). Let Ω be an open connected subset of \mathbb{C} , $f : \Omega \rightarrow \mathbb{C}$ be a continuous function and let $a \in \Omega$. A function $g : \Omega \rightarrow \mathbb{C}$ is a primitive of f if and only if for any $z \in \Omega$ and any rectifiable path γ of Ω that joins a and z ,

$$g(z) = g(a) + \int_{\gamma} f(w) dw.$$

Proof. Let g be a primitive of f and γ be a rectifiable path of Ω that joins a and z . Let $\gamma = \gamma_1 |_{t_1} \dots |_{t_{n-1}} \gamma_n$ where every γ_k is continuously differentiable. For any $k \in \{1, \dots, n\}$, the function

$$\phi : t \in [0, 1] \mapsto g(\gamma_k(t))$$

is differentiable as a composition of real-differentiable functions, with

$$\phi'(t) = dg_{\gamma_k(t)}(\gamma_k'(t)) = g'(\gamma_k(t))\gamma_k'(t).$$

The function ϕ' is continuous, hence, by the real analysis version of the fundamental theorem of calculus, applied to the real and imaginary parts of ϕ' on $]0, 1[$, we have for any positive number ϵ smaller than 1,

$$\phi(1 - \epsilon) - \phi(\epsilon) = \int_{\epsilon}^{1-\epsilon} \phi'(t) dt,$$

hence by continuity of ϕ and ϕ'

$$\phi(1) - \phi(0) = \int_0^1 \phi'(t) dt,$$

which is equivalent to

$$g(\gamma_k(1)) - g(\gamma_k(0)) = \int_0^1 g'(\gamma_k(t))\gamma_k'(t) dt = \int_{\gamma_k} f(w) dw.$$

The sum of these equations for all $k \in \{1, \dots, n\}$ provides

$$g(z) - g(a) = \int_{\gamma} f(w) dw.$$

Conversely, assume that g satisfies the theorem property. Let γ be a rectifiable path of Ω that joins a and z and let $r > 0$ be such that the open disk centered

at z with radius r is included in Ω . Consider the concatenation μ of γ and of the oriented segment $[z \rightarrow z + h]$ for h such that $|h| < r$. It is a rectifiable path of Ω , hence

$$\begin{aligned} g(z + h) &= g(a) + \int_{\mu} f(w) dw \\ &= g(a) + \int_{\gamma} f(w) dw + h \int_0^1 f(z + th) dt \\ &= g(z) + h \int_0^1 f(z + th) dt \end{aligned}$$

hence

$$\frac{g(z + h) - g(z)}{h} = \int_0^1 f(z + th) dt.$$

The right-hand side of this equation converges to $f(z)$ by continuity when h goes to zero, therefore g is a primitive of f . ■

Corollary – Existence of Primitives [†]. Let Ω be an open connected subset of \mathbb{C} . The function $f : \Omega \rightarrow \mathbb{C}$ has a primitive if and only if it is continuous and for any closed rectifiable path γ

$$\int_{\gamma} f(z) dz = 0.$$

Proof – Existence of Primitives. If the function f has primitives, it is the derivative of a holomorphic function, thus it is continuous. Additionally, for any closed rectifiable path γ of Ω , the fundamental theorem of calculus provides

$$g(\gamma(1)) = g(\gamma(0)) + \int_{\gamma} f(w) dw,$$

hence as $\gamma(1) = \gamma(0)$,

$$\int_{\gamma} f(w) dw = 0.$$

Conversely, assume that any such integral is zero. Select any a in Ω and define for any point z in Ω and any rectifiable path γ of Ω that joins them the function

$$g(z) = g(a) + \int_{\gamma} f(w) dw.$$

This definition is non-ambiguous: if we select a different path μ , the difference between the right-hand sides of the definitions would be

$$\left(g(a) + \int_{\gamma} f(w) dw \right) - \left(g(a) + \int_{\mu} f(w) dw \right) = \int_{\gamma | \mu^{\leftarrow}} f(w) dw = 0$$

as $\gamma | \mu^{\leftarrow}$ is a closed rectifiable path of Ω . Consequently, g is uniquely defined and by the fundamental theorem of calculus, it is a primitive of f . ■

Corollary – Set of Primitives. Let Ω be an open connected subset of \mathbb{C} and let $f : \Omega \rightarrow \mathbb{C}$. If $g : \Omega \rightarrow \mathbb{C}$ is a primitive of f , the function $h : \Omega \rightarrow \mathbb{C}$ is also a primitive of f if and only iff it differs from g by a constant.

Proof. It is clear that a function h that differs from g by a constant is a primitive of f . Conversely, if g and h are both primitives of f , $g - h$ is a primitive of the zero function. The fundamental theorem of calculus shows that for any a and z in Ω and any rectifiable path γ of Ω that joins them,

$$g(z) - h(z) = g(a) - h(a) + \int_{\gamma} 0 \, dw = g(a) - h(a)$$

hence their difference is a constant. ■

Corollary – Integration by Parts [†]. Let Ω be an open connected subset of \mathbb{C} and let γ be a rectifiable path of Ω . For any pair of holomorphic functions $f : \Omega \rightarrow \mathbb{C}$ and $g : \Omega \rightarrow \mathbb{C}$,

$$\int_{\gamma} f'g(z) \, dz = [fg(\gamma(1)) - fg(\gamma(0))] - \int_{\gamma} fg'(z) \, dz.$$

Proof. The derivative of the holomorphic function fg is $f'g + fg'$. It is continuous as a sum and product of continuous functions, thus the fundamental theorem of calculus provides

$$fg(\gamma(1)) = fg(\gamma(0)) + \int_{\gamma} (f'g + fg')(z) \, dz,$$

which is equivalent to the conclusion of the corollary. ■

Remark & Definition – Variation of a Function on a Path. The difference between the value of a function f at the terminal value and at the initial value of a path γ may be denoted $[f]_{\gamma}$. With this convention, the formula that connects a function f and its primitive g is

$$[g]_{\gamma} = \int_{\gamma} f(z) \, dz$$

and the integration by parts formula becomes

$$\int_{\gamma} f'g(z) \, dz = [fg]_{\gamma} - \int_{\gamma} fg'(z) \, dz.$$

Appendix – A Better Theory of Rectifiability

Rectifiable Paths

The definition we used so far for “rectifiable” is a conservative one. In this section, we come up with a more general definition of the concept that still meets the requirements for the definition of line integrals.

To “rectify” a path (from Latin *rectus* “straight” and *facere* “to make”) is to straighten – or by extension to compute its length, which is a trivial operation once a path has been straightened.

The general definition of the length of a path does not require line integrals. Instead, consider any partition (t_0, \dots, t_n) of the interval $[0, 1]$ and the path $\mu = \mu_1 |_{t_1} \dots |_{t_{n-1}} \mu_n$ where

$$\mu_k(t) = (1 - t)\gamma(t_{k-1}) + t\gamma(t_k).$$

We may define the length of such a combination of straight lines as

$$\ell(\mu) = \sum_{k=1}^{n-1} |\gamma(t_{k+1}) - \gamma(t_k)|.$$

As the straight line is the shortest path between two points, this number should provide a lower bound of the length of γ . On the other hand, using finer partitions of the interval $[0, 1]$ should also provide better approximations of the length of γ . Following this idea, we may *define* the length of γ as the supremum of the length of μ for all possible partitions of $[0, 1]$:

$$\ell(\gamma) = \sup \left\{ \sum_{k=0}^{n-1} |\gamma(t_{k+1}) - \gamma(t_k)| \mid n \in \mathbb{N}^*, t_0 = 0 < \dots < t_n = 1 \right\}$$

Not every path has a finite length; those who have are by definition *rectifiable*. In general, a function $\gamma : [0, 1] \mapsto \mathbb{C}$ whose length is finite – even if it is not continuous – is of *bounded variation*.

The Line Integral

To define line integrals along the path γ , it is enough that γ is of bounded variation. For any such function γ , we may build a (complex-valued, Borel) measure on $[0, 1]$ denoted $d\gamma$. This measure is defined by its integral of any continuous function $\phi : [0, 1] \rightarrow \mathbb{C}$, as a limit of Riemann(-Stieltjes) sums

$$\int_{[0,1]} \phi d\gamma = \lim \sum_{m=0}^{n-1} \phi(t_m)(\gamma(t_{m+1}) - \gamma(t_m)).$$

The limit is taken over the partitions of the interval $[0, 1]$ with

$$\max \{|t_{m+1} - t_m| \mid m \in \{0, \dots, n-1\}\} \rightarrow 0.$$

The line integral of a continuous function $f : \gamma([0, 1]) \rightarrow \mathbb{C}$ is then defined by

$$\int_{\gamma} f(z) dz = \int_{[0,1]} (f \circ \gamma) d\gamma.$$

The *total variation* $|d\gamma|$ of $d\gamma$ is the positive measure defined by

$$|d\gamma|(A) = \sup_{\mathfrak{P}} \sum_{B \in \mathfrak{P}} |d\gamma(B)|$$

where the supremum is taken over all finite partitions \mathfrak{P} of A into measurable sets. This measure provides an integral expression for the length of γ :

$$\ell(\gamma) = \int_{[0,1]} |d\gamma|.$$

A Non-Rectifiable Curve

The Koch snowflake (Koch 1904) is an example of a continuous curve which is nowhere differentiable; it is also a non-rectifiable closed path. It is defined as the limit of a sequence of polylines γ_n . The first element of this sequence is an oriented equilateral triangle:

$$\gamma_1 = [0 \rightarrow 1 \rightarrow e^{i\pi/3} \rightarrow 0].$$

Then, γ_{n+1} is defined as a transformation of γ_n : every oriented line segment $[a \rightarrow a + h]$ that composes γ_n is replaced by the polyline:

$$\left[a \rightarrow a + \frac{h}{3} \rightarrow a + \left(1 + e^{-i\pi/3}\right) \frac{h}{3} \rightarrow a + 2\frac{h}{3} \rightarrow a + h \right]$$

The Koch snowflake γ is defined as the limit of the γ_n sequence. The geometric construction yields that for any n greater than zero,

$$\forall t \in [0, 1], |\gamma_{n+1}(t) - \gamma_n(t)| \leq \left(\frac{1}{3}\right)^n \frac{\sqrt{3}}{2}.$$

As $\sum_{p=0}^{+\infty} \left(\frac{1}{3}\right)^p = \frac{1}{1-1/3} = \frac{3}{2}$, for any positive integer p we have

$$\forall t \in [0, 1], |\gamma_{n+p}(t) - \gamma_n(t)| \leq \left(\frac{1}{3}\right)^n \frac{3\sqrt{3}}{2}.$$

The sequence γ_n is a Cauchy sequence in the space of continuous and complex-valued functions defined on $[0, 1]$; its uniform limit exists and is also continuous.

On the other hand, the curve is not rectifiable. First, the definition of the sequence γ_n makes it plain that every iteration increases the initial length of the path by one-third:

$$\ell(\gamma_n) = 3 \times \left(\frac{4}{3}\right)^{n-1}.$$

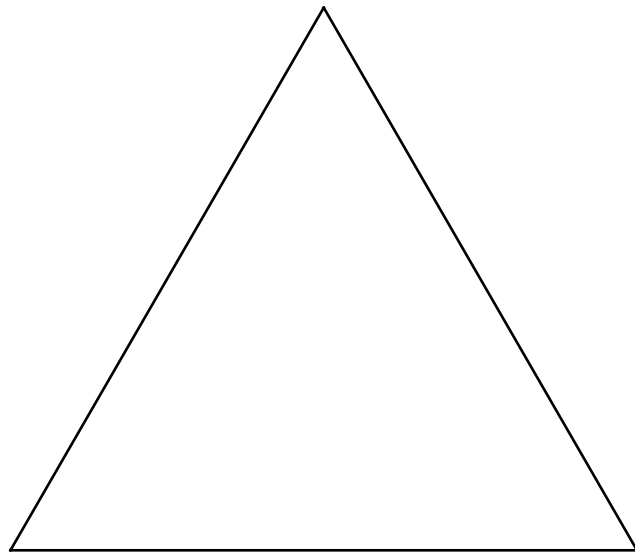


Figure 3: Image of the Koch snowflake, first iteration.

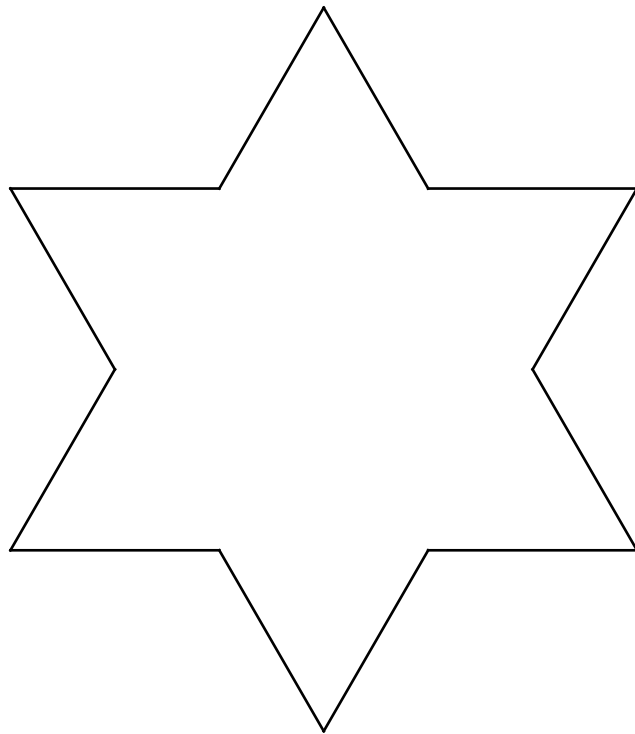


Figure 4: Image of the Koch snowflake, second iteration.

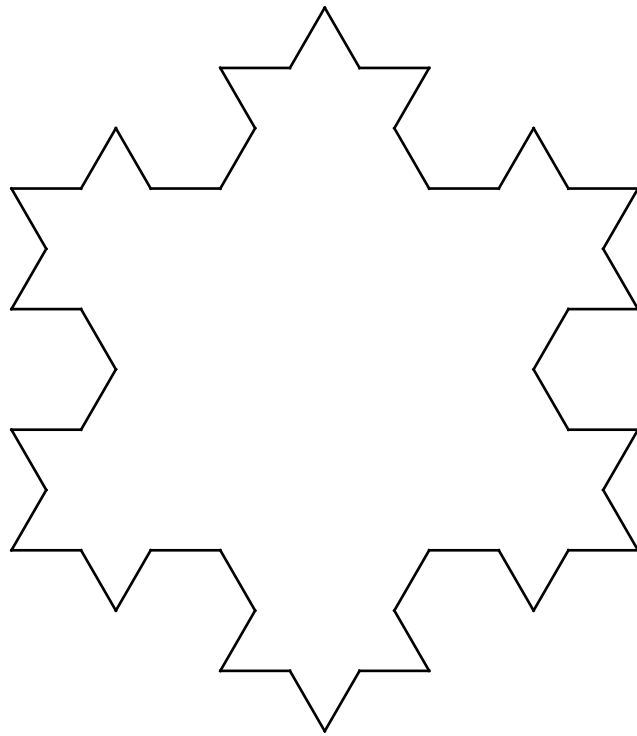


Figure 5: Image of the Koch snowflake, third iteration.

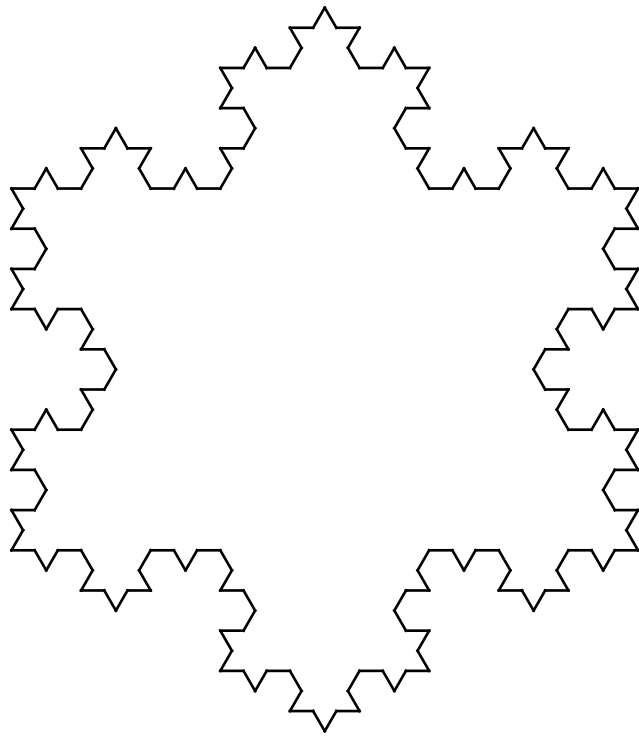


Figure 6: Image of the Koch snowflake, fourth iteration

The length of γ_n tends to $+\infty$ when $n \rightarrow +\infty$. Now, every point at the junction of the segments of the polyline γ_n also belongs to the Koch snowflake; more precisely

$$\forall m \in \{0, \dots, 3 \times 4^{n-1}\}, \gamma\left(\frac{m}{3 \times 4^{n-1}}\right) = \gamma_n\left(\frac{m}{3 \times 4^{n-1}}\right).$$

Therefore

$$\ell(\gamma) \geq \sum_{m=0}^{3 \times 4^{n-1} - 1} \left| \gamma\left(\frac{m+1}{3 \times 4^{n-1}}\right) - \gamma\left(\frac{m}{3 \times 4^{n-1}}\right) \right| = \ell(\gamma_n)$$

and thus $\ell(\gamma) = +\infty$: the path γ is not rectifiable.

References

Koch, Helge von. 1904. "Sur une courbe continue sans tangente, obtenue par une construction géométrique élémentaire." *Arkiv för Matematik, Astronomi och Fysik* 1. Kungliga Svenska Vetenskapsakademien.: 681–702.